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*PIGMENTS FROM MARINE MUDS**

BY DENIS L. FOX AND LLOYD J. ANDERSON

SCRIPPS INSTITUTION OF OCEANOGRAPHY

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Earlier investigations in this laboratory[1] revealed the presence of carotenoids, along with greenish pigments presumably representing the breakdown of chlorophylls, in muds from the ocean floor.

An opportunity for studying such pigments more extensively and in greater quantities has been provided recently through the kindness of Dr. R. R. Revelle and his associates at the Scripps Institution, whose oceanographic and geological researches in the Gulf of California were supported by a grant from the Geological Society of America.

The mud core sections placed at our disposal were taken from the sea floor near the middle of the Gulf, at latitudes between 27° 17.2′ and 27° 53.6′, longitudes between 110° 41.5′ and 111° 7.9′, at depths ranging between 201 and 1750 meters below the water-surface and from sediment layers between 8 inches and 17 feet beneath the surface of the bottom. The muds were free from oxygen; indeed many of them evolved either H_2S or an inflammable gas (CH_4 and H_2) when first brought up. Samples were stored in sealed tins at low temperatures until the time of analysis. Pigments were extracted from mud samples by exhaustive treatment with acetone, transferred to petroleum ether and resolved into two gross primary fractions, i.e., preferentially soluble in either the latter solvent or 90 per cent methanol. Chromatographic adsorption on Tswett columns of $CaCO_3$ removed all of the several greenish pigments from the methanol-free epiphasic petroleum ether solution, while yellow or orange pigments of the carotene class were similarly separated on layers of $Ca(OH)_2$ below. Hypophasic fractions (in 90 per cent methanol) were generally quite free of green pigments and contained only oxygenated carotenoids of the xanthophyll group. After transference to pure petroleum ether or benzene, they were separable into chromatographic fractions on $CaCO_3$ columns.

Studies of differences in solubility, chromatographic adsorption, behavior toward treatment with alcoholic alkali, and types of absorption spectra

have revealed the presence of a series of greenish pigments (presumably derived from chlorophyll breakdown) and numerous epiphasic and hypophasic carotenoids.

Green Pigments.—Three principal fractions were separated, one soluble in petroleum ether and resolvable into two chromatographic zones, another insoluble in petroleum ether but soluble in acetone and a third insoluble in petroleum ether with or without acetone, but soluble in aqueous acetone. Each pigment showed from three to four absorption bands in the visible spectrum, occupying positions from the blue-green (507 mμ) to the red (683 mμ). Treatment of any of the pigments with alcoholic KOH caused some breakdown, resulting in colored bodies with simpler spectra, but always showing, like the parent substances, a prominent absorption maximum in the red and general increasing absorption in the violet and shorter wavelength regions. A prominent pigment occurring in the petroleum ether fraction of all extracts showed an absorption spectrum very like that recorded by Rothemund[2] for hemp-speed protochlorophyll, its maxima in ethyl ether being at 507, 537, 611 and 667 mμ, and in chloroform at 510, 546, 630 and 682 mμ. Incineration yielded no ash.

Chlorophylls themselves were never encountered, but the other green compounds were found in quantity at all depths studied. Assuming that these green pigments are decomposition products of chlorophylls from diatoms and other marine algae, the recovery of striking quantities of a pigment closely resembling protochlorophyll from ancient, strongly reducing marine deposits reëmphasizes questions pondered by plant biochemists regarding biochemical relationships between protochlorophyll and chlorophyll.

Carotenoids.—Most of these pigments remained in the epiphase. The additional facts that they remained epiphasic to 90 or even 95 per cent methanol after treatment with hot alcoholic KOH and that they were unadsorbed by $CaCO_3$ placed them in the carotene class. A predominant pigment of this group showed absorption maxima agreeing closely with those of β-carotene. α-Carotene was represented in smaller amounts, while other pigments were encountered which exhibited two or three absorption bands not listed in the literature, but not very unlike rhodopurpurin extracted from the pigmented bacterium *Rhodovibrio*. A pigment with absorption maxima close to those of torulene from the red yeast, *Torula rubra*, was found in several samples. On one occasion, hydrolysis of the torulene zone from the chromatogram yielded an acidic carotenoid with a single absorption maximum at about 500 mμ in pyridine. These properties, with its deep rose color in CS_2, likened it to astacene, encountered in certain invertebrates.

The hypophasic fractions included pigments showing absorption maxima agreeing closely with those of fucoxanthin (found in many marine algae),

lutein and zeaxanthin. There were numerous xanthophylls with unfamiliar absorption spectra. No acidic carotenoids were found in any of the hypophasic fractions.

While quantitative estimations of pigments have not yet been correlated with depth of mud, such work is anticipated. However an account of a preliminary analysis made of one sample will be of interest. One gulf-bottom core taken at lat. 27° 41′, long. 110° 55′ at a depth of 666 m. was 17 feet in length and yielded an inflammable gas from its deeper layers. Two hundred fifty grams of mud, taken from the 15 to 16.3 foot level yielded 5.22 mg. of epiphasic carotenoids (in equivalents of β-carotene with whose absorption spectrum that of the mixed pigments was very similar) and 2.33 mg. of hypophasic carotenoids (in equivalents of lutein, whose absorption spectrum was similar) giving a figure of about 3 mg. carotenoids per 100 grams wet mud, or some 6 mg. per cent on a dry basis.

Such quantities of natural pigments, chemically unstable under ordinary conditions, would seem to be rather remarkable when one considers the probable age of this deep layer of mud in the gulf, estimated by Dr. Revelle as being some 6000 to 7000 years (personal communication). The absence of living photosynthetic plants and the extreme paucity of active forms of other microörganisms would suggest that we are dealing here with fragmentary biochemical chapters of past ages. The pigmentary record in buried marine muds is probably altered but little with the passage of time, due to stable conditions of darkness, low temperatures, little if any biological activity and especially the absence of oxygen.

Gulf muds yielded pigmentary compounds in greater quantity than did bottom-samples examined from the open ocean. No obvious differences were noted in quantity of pigment extractable relative to depth below the mud surface. It is expected that quantitative data will be available later.

Other Pigments.—While no such substances as the one about to be described were found in any of the samples of gulf mud examined, one non-carotenoid fluorescent pigment attracting special interest occurred in a mud core taken from the open sea (about lat. 32° 30′, long. 117° 30′) at a depth of some 1300 meters. Samples from the first six inches and from the 6 and 7 foot mud-depth yielded green pigments and carotenoids, but a section from the 40 to 46 inch level yielded, along with the usual green pigment (absorption maximum 669 mμ in petroleum ether from all four core-sections examined), no carotenoids, but instead a deep brown-yellow pigment, showing pronounced green fluorescence in petroleum ether solution.

A pigment obtained from a sample of heavy crude petroleum from Kern County† showed striking similarities to the one just mentioned. Each pigment was dark, amorphous and tarry, preferentially soluble in petroleum ether over 90 per cent methanol, sparingly soluble in pure methanol, showed blue-green fluorescence in petroleum ether or chloroform, and

passed through $CaCO_3$ in benzene solution, showing a diffuse brown zone at the upper end of the $Ca(OH)_2$ column. Further, a series of three chromatograms, involving (1) the mud pigment, (2) the oil pigment and (3) a mixture of these two, gave a similar, single zone of brown sorbed material in each tube (the so-called "mixed chromatogram" used in testing suspected identical compounds). Finally, each pigment gave the same absorption maxima, at 575 mμ and 553 to 554 mμ.

These comparative observations and tests indicate the identity or close chemical relationship between a pigment from marine mud and one separated from crude petroleum. Dr. Revelle's estimate of the age of the deposit of mud at the depth from which the pigment was obtained is about 4000 years (personal communication).

Résumé.—The demonstration of numerous chlorophyll derivatives and carotenoids in ancient marine deposits is thought-provoking. Chlorophylls are known to undergo advanced biochemical changes in animal organisms, while carotenoids found in feces may retain their complete integrity. The absence of chlorophylls themselves in the presence of a suspected protochlorophyll and other simpler degradation products leads to a consideration of possible biochemical changes wrought by non-chlorophyllous microorganisms, and emphasizes the questions regarding interrelationships between chlorophylls and protochlorophyll in green plants.[2]

The encountering of relatively rich deposits of numerous carotenoids also bears on the question of metabolic activities in marine muds. Does the preponderance of carotene-like pigments over xanthophylls in gulf muds actually indicate an excess of fungal and bacterial over algal remains, or have isomerization and other chemical changes taken place, with or without biological intervention, to alter so extensively the situation encountered in marine algae?

The question of synthesis of carotenoids by bacteria and fungi isolated from marine muds is being investigated, and problems involving possible changes in chlorophyll and certain carotenoids by such organisms are being undertaken.

The encountering of a fluorescent, non-carotenoid pigment in both a sample of crude petroleum and a marine mud core (accompanied in the latter by a green pigment, supposedly resulting from chlorophyll degradation) lays emphasis upon the importance of continuing investigations on the biogenic origins of petroleum.

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† This sample, kindly furnished by Mr. R. A. Halloran, Manager of the Research and Development Department of the Standard Oil Company of California, is designated by Mr. Halloran as originating in marine formations of upper Miocene or lower Pliocene age (Sample number 25292-R).

[1] Fox, D. L., *Proc. Nat. Acad. Sci.*, **23**, 295–301 (1937).

[2] Rothemund, P., *Cold Spring Harbor Monogr. Quant. Biol.*, **3**, 71–79 (1935).

FEMALE STERILITY IN INTERRACIAL HYBRIDS OF DROSOPHILA PSEUDOÖBSCURA

BY KLAUS MAMPELL

WM. G. KERCKHOFF LABORATORIES, CALIFORNIA INSTITUTE OF TECHNOLOGY

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Male sterility is a regular occurrence in the hybrids of race *A* and race *B* of *D. pseudoöbscura*.[1] A case of female sterility in the same hybrids has been described recently.[2] The genetic analysis of this sterility will here be extended, and new data concerning the map of the *X*-chromosome will be given.

The salivary chromosomes of the *A B* hybrid female show two inversions in the *X*-chromosome, one in the right, one in the left arm. The *XR* inversion has now been established genetically. In race *A* the genetical map shows the genes: compressed, sepia, ascute, javelin (slender) in the order given.[3] The *X*-chromosome of race *B*, based on the map already known[4] and on additional data, shows the order: compressed, ascute, sepia, javelin (table 1). A comparison of the maps shows that the breakpoints of the inversion lie to the right of compressed and to the left of javelin, respectively.

Included in the map are some genes which seem to have no counterpart in either race *A* or *D. melanogaster*. Filiform is a good dominant viable in male and homozygous female; the macrochaetae are long and slender with tapering tips. The lethal plexus kills the males in the late pupal stage; when occasional males emerge their wings show a plexus-like effect; they are very weak and die soon after emergence. The mutant radius rudimentary is expressed in the female only. The suppression of the effect in the male is not due to the influence of the *Y*-chromosome (tested in *XO* males). The expression in the female is rather variable and is influenced by environmental conditions; in flies raised at a temperature of 15°C. very

marked plexus effects are added to the rudimentary condition of the radius; at 25°C. the character usually overlaps wild type.

In race *A*, Gottschewski[2] has described two genes, lm_1 and lm_2, which have a certain lethal action in an environment of hybrid cytoplasm. In race *B* there are two corresponding genes which have a similar action. The symbols for these genes, proposed here, are: lm_1A and lm_2A for race *A*, lm_1B and lm_2B for race *B*.

Certain backcrosses of hybrid females were found to produce no or very few offspring. An example of certain mass matings will illustrate this: 30 males of a wild race *A* strain (Boise 1) were mated to 20 hybrid females with a vermilion sepia *X*-chromosome from the *B* father, an eosin vermilion *X*-chromosome from the *A* mother. Since the hybrid females were homozygous for vermilion, all normal male offspring should have carried that gene. The total offspring of these 20 females consisted of 47 wild-type

TABLE 1

THE *X*-CHROMOSOME OF RACE *B*

beaded	0.0	rose	38.9
scutellar	24.9	bubble	53.3
yellow	24.9	fused	54.0
Notch		compressed	
white	31.3	ascute	55.7
lm_1B*		scarlet	76.1
singed	32.7	lm_2B*	
vermilion	34.5	sepia	105.8
Filiform	34.6	radius rudimentary	128.2
lethal plexus	34.8	javelin	133.0
dusky	35.3	short	138.6
forked	35.8		

* The position of the interracial lethals is uncertain.

females and 15 wild-type males. These patroclinous males could not have inherited their *X*-chromosome from the mother, but must have resulted from the union of an *X*-bearing sperm and a no-*X* egg. This no-*X* egg can arise through primary non-disjunction or, as seems more likely, through crossing-over within the inversions, a four strand double crossover producing only dicentric and acentric *X*-chromosomes.

Gottschewski pointed out that the lethals are not completely dominant in the backcross females; in many cases "rare survivors" appear. The 47 females from the above cross belong in that category.

When the backcross males were w^e *v* (*A*) or *v se* (*B*), no offspring at all were produced; this is expected, since the *XO* males too, in this case, would inherit the lethals. In a third type of cross: a hybrid female free of the

lethals, backcrossed to a male carrying them should produce males and no or few females; the data bear out this expectation.

From these typical examples it may be concluded that both the *w^f v* (*A*) *X* and the *v se* (*B*) *X* must have carried lethal genes which were completely innocuous intraracially; also, they did not affect the F_1 hybrid generation where they would still be in an environment of race *A* or race *B* cytoplasm. It is the hybrid cytoplasm, then, which allows these genes to act as lethals.

Two genes must be concerned in the production of the lethal effect as shown previously by Gottschewski. A race *A* fly which is known to carry the lethals and which has the proper marker genes in the *X* is mated to a race *B* fly which is known to be free of the lethals. The resulting hybrid female, when backcrossed, gives a majority of wild-type sons. Those sons carrying any race *A* marker genes prove to be the result of crossing-over with the race *B X*-chromosome. No sons appear which have intact the section between the white region and the *XR* inverted region. Those males survive in which the two regions have become separated by crossing-over. Since crossovers in both directions survive, lm_1A in the white region and lm_2A in the *XR* inverted region must be together to produce the lethal effect; individually they no longer act as lethals. The reciprocal mating shows a corresponding situation in race *B*; lm_1B and lm_2B lie in the same respective regions and act in the same manner as the race *A* lethals.

The maternal effect by which these interracial lethals act was investigated by the experiment shown in table 2. A race *B* wild-type male is mated to a race *A* female which carries in the *X*-chromosome an inversion across the spindle attachment region associated with a dominant effect, short scutellum (*Ss*); this *X*-chromosome is lethal to the male. The autosomes are marked as follows: chromosome II carries Bare (*Ba*) in an x-ray induced inversion; chromosome III carries Blade (*Bl*) in standard sequence; chromosome IV carries Curly (*Cy*) in an x-ray induced inversion; chromosome V cannot be marked. In the hybrid female crossing-over should be suppressed effectively and the identity of racial origin of each chromosome be preserved, since in addition to the inversions mentioned above, there are present two interracial inversions in *X*, one in II. Crossing-over is suppressed least effectively in III, since some crossing-over does occur between Standard and Klamath, the common race *B* sequence. *Ss Ba Bl Cy* hybrid females are backcrossed to *w^f v* males of race *A*; the *w^f v* strain carries the race *A* interracial lethals. From this cross only few female offspring are obtained; they consist of the 16 different phenotypes given in table 2. Each phenotype individually is backcrossed to *w^f v*. All classes with the *Ss* marker are fertile; all classes without the *Ss* marker are sterile. All the fertile females are homozygous for the race *A X*-chromosome; all sterile females are racially heterozygous for *X*. The race *A* interracial lethals act only in cytoplasm (eggs) produced by a female

TABLE 2

$+(B)\ ♂ \times Ss\ Ba\ Bl\ Cy\ (A)\ ♀$

$w^e\ v\ (A)\ ♂ \times \underline{Ss}\ \underline{Ba}\ \underline{Bl}\ \underline{Cy}\ ♀ \times v\ dy\ (B)\ ♂$

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$w^e\ v\ (A)\ ♂ \times$	$w^e\ v\ (A)$	$+(A)$	$+(A)$	$+(A)$		$v\ dy\ (B)\ ♂ \times$	$v\ dy\ (B)$	$+(B)$	$+(B)$	$+(B)$	
	Ss	Ba	Bl	Cy	$= f$		Ss	Ba	Bl	Cy	$= *(s)$
	$+$	$+$	$+$	$+$	$= s$		$+$	$+$	$+$	$+$	$= f(13)$
	Ss	$+$	$+$	$+$	$= f$		Ss	$+$	$+$	$+$	$= f(5)$
	$+$	Ba	Bl	Cy	$= s$		$+$	Ba	Bl	Cy	$= *(s)$
	Ss	Ba	$+$	$+$	$= f$		Ss	Ba	$+$	$+$	$= f(13)$
	$+$	$+$	Bl	Cy	$= s$		$+$	$+$	Bl	Cy	$= s(.5)$
	Ss	Ba	Bl	$+$	$= f$		Ss	Ba	Bl	$+$	$= *(f)$
	$+$	$+$	$+$	Cy	$= s$		$+$	$+$	$+$	Cy	$= s(3)$
	Ss	$+$	Bl	Cy	$= f$		Ss	$+$	Bl	Cy	$= s(0)$
	$+$	Ba	$+$	$+$	$= s$		$+$	Ba	$+$	$+$	$= f(11)$
	Ss	$+$	$+$	Cy	$= f$		Ss	$+$	$+$	Cy	$= s(0)$
	$+$	Ba	Bl	$+$	$= s$		$+$	Ba	Bl	$+$	$= *(f)$
	Ss	Ba	$+$	Cy	$= f$		Ss	Ba	$+$	Cy	$= s(1)$
	$+$	$+$	Bl	$+$	$= s$		$+$	$+$	Bl	$+$	$= f(6)$
	Ss	$+$	Bl	$+$	$= f$		Ss	$+$	Bl	$+$	$= *(f)$
	$+$	Ba	$+$	Cy	$= s$		$+$	Ba	$+$	Cy	$= s(0)$

In the 16 classes of backcross females all marked chromosomes are race A, all $+$ chromosomes are race B. In the $w^e\ v$ backcross, s = sterile, f = fertile. In the $v\ dy$ backcross, s = no or few $v\ dy$ females produced, $f = v\ dy$ females produced, * = class not obtained, symbol in parenthesis refers to expectation. Numbers in parenthesis indicate average $v\ dy$ ♀ offspring per mother.

racially heterozygous for the *X*-chromosome. The autosomes have little or no influence.

When *Ss Ba Bl Cy* hybrid females are backcrossed to *v dy* (*B*) males which carry the race *B* interracial lethals, a somewhat different picture presents itself. Of the 16 different phenotypes only eleven have been obtained; all of these produce some offspring. In deciding which chromosome combinations are responsible for the lethality of the lm_1B lm_2B complex the male offspring are neglected, first, because the viability is decreased by certain *Y*-chromosomes,[5] an uncontrolled factor, and, second, because the results would be obscured by independent sex-linked lethals whose mutation rate is increased by hybridization.[6] The number of *v dy* female offspring per mother is used for the determination of the action of the interracial lethals, since only this kind of progeny is comparable in all classes. In table 2 it can be seen that none of the classes with the *Cy* marker show as many *v dy* females per mother as any of the classes without the *Cy* marker. It seems, therefore, that the race *B* interracial lethals act in cytoplasm (eggs) produced by a female racially heterozygous for chromosome IV. It must be pointed out, however, that the results are not nearly as clear as those for the race *A* lethals; caution must be used in accepting chromosome IV as the only one responsible, until further evidence is obtained.

With these interracial lethals, dominant in hybrid cytoplasm, an effective isolating mechanism is accomplished, since no or very few offspring are obtained from hybrid females which are backcrossed to any kind of males. The individuals carrying the lethals die in the egg or early larval stage. Whether or not this mechanism in fact is employed in nature will become evident from an analysis of wild populations which is now in progress.

Summary.—1. The break-points of the *XR* interracial inversion lie to the right of compressed and to the left of javelin, respectively.

2. The genetic isolation of race *A* and race *B* has progressed an additional step in the hybrids of certain strains where the females produce no or very few offspring. This is accomplished through interracial lethal genes connected with a maternal effect.

[1] Lancefield, D. E., *Z. i. A. V.*, **52**, 287–317 (1929).
[2] Gottschewski, G., *Ibid.*, **78**, 338–398 (1940).
[3] Sturtevant, A. H., and C. C. Tan, *Jour. Genet.*, **34**, 415–432 (1937).
[4] Beers, C. V., *Genetics*, **22**, 577–586 (1937).
[5] Sturtevant, A. H., *Proc. Nat. Acad. Sci.*, **23**, 360–362 (1937).
[6] Sturtevant, A. H., *Ibid.*, **25**, 308–310 (1939).

GROUPS CONTAINING MAXIMAL SUBGROUPS OF PRIME ORDER

BY G. A. MILLER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

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A maximal subgroup of a group G is a proper subgroup of G which with any operator of G not contained in this subgroup generates G. Every operator of G which does not generate G therefore appears in at least one maximal subgroup of G. A fundamental case presents itself when this maximal subgroup is of prime order p and we begin with a study of this case. When G is abelian this case is very simple since G is then evidently either one of the two possible groups of order p^2 or the cyclic group of order pq, where p and q are distinct prime numbers. In what follows we may therefore confine our attention to the case when G is a non-abelian group which contains a maximal subgroup of prime order and this will be done unless the contrary is explicitly stated.

Suppose now that G contains a maximal subgroup of order 2 generated by the operator s. If t represents any operator of G which is not cummutative with s the two operators s and $t^{-1}st$ are both of order 2 and hence they generate the dihedral group whose order is twice the order of their product. Since s is a maximal subgroup of G this dihedral group is G itself and the order of G is twice an odd prime number since otherwise s would not generate a maximal subgroup of G. Moreover, every such dihedral group contains a maximal subgroup of order 2. Hence there results the following theorem: *A necessary and sufficient condition that a non-abelian group contains a maximal subgroup of order 2 is that it is the dihedral group whose order is twice an odd prime number.*

From the preceding paragraph it results that when G contains a maximal subgroup of order 2 it is non-invariant and G contains also another maximal subgroup of prime order which is invariant under G. The general case when G contains an invariant maximal subgroup of prime order is very simple since G then contains also a non-invariant maximal subgroup of prime order. In fact, if the former maximal subgroup is of order p the order of the latter is a divisor of p-1, where p is an odd prime number. Every prime divisor of p-1 gives rise to a group which has maximal subgroups of two distinct prime orders and we proceed to prove that this includes all the non-abelian groups which contain maximal subgroups of as many as two distinct prime orders. It was noted above that the only abelian group which contains maximal subgroups of as many as two distinct prime orders is the cyclic group of order pq, p and q being distinct prime numbers.

Suppose that G contains a non-invariant maximal subgroup of order p. If the order of G is not a power of p this subgroup must appear in a Sylow subgroup whose order is a power of p according to the well-known theorems that every subgroup whose order is a power of p appears in a Sylow subgroup whose order is a power of p whenever the order of the group is not a power of p, and a group whose order is a power of p cannot contain a non-invariant maximal subgroup of order p. Hence G contains Sylow subgroups of order p and as these are maximal they are transformed under G according to a primitive permutation group whose permutations, besides the identity, which omit one letter, omit exactly one letter. Hence G contains an invariant subgroup of index p generated by its permutations of highest degree according to a well-known theorem due to G. Frobenius (1849–1917).

If this invariant subgroup is of prime order these groups come under the case considered above and hence we may confine our attention here to the case when this subgroup is of composite order. Since G is non-abelian the number of its operators of order p is equal to the order of G multiplied by $(p\text{-}1)/p$. As this is more than half the order of G when p is odd it has been proved that if a non-abelian group contains maximal subgroups of at least two distinct prime orders then the order of this group is pq, p and q being prime numbers such that p-1 is divisible by q. The given invariant subgroup of index p contained in G is a regular permutation group if G is represented on letters corresponding to the given subgroups of order p and hence its order is $1 + kp$. In particular, *a simple group of composite order cannot contain a maximal subgroup of prime order.*

It remains only to determine the possible groups which separately contain maximal subgroups of a single prime order p and in which the invariant subgroup of order $1 + kp$ is not of prime order. The tetrahedral group is an instance of such a group. In this case $p = 3$ and $k = 1$. Another instance is furnished by the group of order 56 which contains eight subgroups of order 7. In this case $p = 7$ and $k = 1$. In both of these cases the invariant subgroup of index p and of order $1 + kp$ is abelian and is composed of operators of the same order besides the identity. It is easy to see that whenever this subgroup is abelian all of its operators besides the identity are of the same order since otherwise this invariant abelian subgroup would involve a characteristic subgroup besides the identity which with a subgroup of order p contained in G would generate a proper subgroup of G. This is impossible since the subgroups of order p are maximal.

When the invariant subgroup of order $1 + kp$ contained in G is non-abelian it must coincide with its commutator subgroup since otherwise this commutator subgroup and a subgroup of order p contained in G would generate a proper subgroup of G. In particular, this invariant subgroup cannot be a solvable non-abelian group. If its order is a power of a prime

number it must be abelian and hence all of its operators besides the identity must have the same order. A necessary and sufficient condition that the order of the given invariant subgroup of order $1 + kp$ is a power of a prime number is that the order of G is divisible by two and only two distinct prime numbers and a necessary and sufficient condition that this invariant subgroup is abelian is that all of its operators besides the identity are of the same order. The operators of order p contained in G appear in p-1 sets of conjugates since G contains an invariant subgroup of index p.

The maximal subgroups of order p contained in G are also minimal subgroups of G. It is not difficult to determine all the groups in which every subgroup which is not maximal is minimal and vice versa. It is known that if all the proper subgroups of a group are maximal then all of these subgroups are also minimal and vice versa (these PROCEEDINGS, **27**, 68 (1941)). If the order of a group is divisible by p^3 but exceeds p^3 its subgroups of order p^2 are neither maximal nor minimal and if the order of a group is not divisible by the square of a prime number but by at least three distinct prime numbers the subgroups whose orders are the product of the two largest of these prime numbers are neither maximal nor minimal when the number of these prime numbers exceeds three. When it is three these subgroups are maximal but not minimal. In this case every subgroup of G is either maximal or minimal and there is no subgroup in G which is both maximal and minimal.

Suppose that the order of G is divisible by p^2 but not by p^3 and that it contains subgroups of order p^2 but is of a larger order. These subgroups are then conjugate under G since they are Sylow subgroups. As they are maximal they are transformed under G according to a primitive permutation group which contains a regular invariant subgroup as was noted above. In the present case it must be of prime order since G would otherwise involve a proper subgroup which would include a subgroup of order p. Hence this case comes under the case considered above. If the order of a group is divisible by a higher power of p than p^3 its subgroups of order p^2 are neither maximal nor minimal but in a group of order p^3 every subgroup is either maximal or minimal. If the order of a group is divisible by p^3 but is of a larger order than p^3 its subgroups of order p^2 are neither maximal nor minimal. Hence there results the following theorem: *If the order of a group is the product of two prime numbers, equal or unequal, then each of its proper subgroups is both maximal and minimal; if this order is the product of three such prime numbers then each of the subgroups is either maximal or minimal; if this order is the product of more than three such prime numbers then it contains proper subgroups which are neither maximal nor minimal.*

It was noted above that when G contains a maximal subgroup of order 2 it is dihedral. Suppose, now, that G contains a maximal subgroup or order 3 generated by s_1 and that s_2 is in the same co-set as s_1 with respect to the

invariant subgroup of order $1 + 3k$. Since s_1s_2 is of order 3 we have $(s_1s_2)^3 = (s_1s_2{}^{-1}s_1s_2)^3 = 1$. It is easy to prove that $s_1s_2s_1$ generates a cyclic group whose generators are commutative with their conjugates under s_1 and under s_2. That is, if a group contains a maximal subgroup of order 3 it contains an invariant abelian subgroup of index 3 which is either cyclic or of type 1^m. The groups which contain maximal subgroups of order 2 or of order 3 may therefore be regarded as determined, but those which contain maximal subgroups of order 2 are naturally considerably simpler than those which contain maximal subgroups of order 3.

[1] Cf. these PROCEEDINGS **27**, 212–216 (1941).

ON THE RIEMANN HYPOTHESIS IN FUNCTION-FIELDS

BY ANDRÉ WEIL

NEW SCHOOL FOR SOCIAL RESEARCH

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A year ago[1] I sketched the outline of a new theory of algebraic functions of one variable over a finite field of constants, which may suitably be described as transcendental, in view of its close analogy with that portion of the classical theory of algebraic curves which depends upon the use of Abelian integrals of the first kind and of Jacobi's inversion theorem; and I indicated how this led to the solution of two outstanding problems, viz., the proof of the Riemann hypothesis for such fields, and the proof that Artin's non-abelian L-functions on such fields are polynomials. I have now found that my proof of the two last-mentioned results is independent of this "transcendental" theory, and depends only upon the algebraic theory of correspondences on algebraic curves, as due to Severi.[2]

Γ being a non-singular projective model of an algebraic curve over an algebraically closed field of constants, the variety of ordered couples (P, Q) of points on Γ has the non-singular model $\Gamma \times \Gamma$ (in a bi-projective space); correspondences are divisors on this model, additively written; they form a module $\mathfrak{C}$ on the ring Z of rational integers. Let Γ_A, Γ_B', Δ, respectively, be the loci of points (P, A), (B, P) and (P, P), on $\Gamma \times \Gamma$, A and B being fixed on Γ and P a generic point of Γ (in the precise sense defined by van der Waerden[3]). The intersection number (C, D) of C and D being defined by standard processes[4] for irreducible, non-coinciding correspondences C, D, will be defined for any C and D which have no irreducible component in common, by the condition of being linear both in C and in D. The degrees $r(C)$, $s(C)$, and the coincidence number $f(C)$

of C are defined as its intersection numbers with a generic Γ_A, with a generic Γ_B' and with Δ, respectively. Transformation $(P, Q) \to (Q, P)$, which is a birational, one-to-one involution of $\Gamma \times \Gamma$ into itself, transforms a correspondence C into a correspondence which will be denoted by C'. To every rational function φ on $\Gamma \times \Gamma$ is attached a divisor (the divisor of its curves of zeros and poles), i.e., a corresponding C_φ; and $(C_\varphi, D) = 0$ whenever it is defined. Let $\mathfrak{C}_0$ be the module of all correspondences of the form $C_\varphi + \Sigma a_i \Gamma_{A_i} + \Sigma b_j \Gamma_{B_j}'$: if $C \in \mathfrak{C}_0$, then $f(C) = r(C) + s(C)$ (whenever $f(C)$ is defined), since this is true for $C = C_\varphi$ (both sides then being 0) and for $C = \Gamma_A$, $C = \Gamma_B'$. Put $\mathfrak{R} = \mathfrak{C} - \mathfrak{C}_0$; an element γ in $\mathfrak{R}$ is a co-set in $\mathfrak{C}$ modulo $\mathfrak{C}_0$, and it can be shown that there always is a C in this co-set for which $f(C)$, $r(C)$, $s(C)$ are defined, so that, putting $\sigma(\gamma) = r(C) + s(C) - f(C)$, $\sigma(\gamma)$ is defined as a linear function over $\mathfrak{R}$. If γ_0 in $\mathfrak{R}$ corresponds to Δ in $\mathfrak{C}$, $\sigma(\gamma_0) = 2g$, where g is the genus of Γ. If $C \in \mathfrak{C}_0$, we have $C' \in \mathfrak{C}_0$, and so the reciprocal relation $(C \rightleftarrows C')$ in $\mathfrak{C}$ induces a similar relation $(\gamma \rightleftarrows \gamma')$ in $\mathfrak{R}$; and $\sigma(\gamma') = \sigma(\gamma)$.

The product of correspondences now being defined essentially as in Severi, $\mathfrak{C}$ becomes a ring, with Δ as unit-element; $\mathfrak{C}_0$ is two-sided ideal in $\mathfrak{C}$, so that $\mathfrak{R}$ also can be defined as ring, with unit-element γ_0: we write $\gamma_0 = 1$. It is found that the degrees of $C \cdot D$ are $r(C) \cdot r(D)$, $s(C) \cdot s(D)$; and that (C, D) is the same as $(C \cdot D', \Delta) = f(C \cdot D')$, and hence is equal to $r(C) \cdot s(D) + r(D) \cdot s(C) - \sigma(\gamma \cdot \delta')$ if γ, δ are the elements of $\mathfrak{R}$ which correspond to C, D. It follows that $\sigma(\gamma\delta) = \sigma(\delta\gamma)$. From $\sigma(n \cdot 1) = 2ng$ it follows that ring $\mathfrak{R}$ has characteristic 0.

Defining the complementary correspondence to C as in Severi,[5] it is found that the generic complementary correspondence to C is irreducible and has degrees (g, m), where $2m = \sigma(\gamma\gamma')$: it follows that $\sigma(\gamma\gamma') \geqslant 0$, and that $\sigma(\gamma\gamma') = 0$ only if $\gamma = 0$.

Now, let k (as in my note[1]) be the Galois field with q elements; k_n its algebraic extension of degree n; $\bar{k}$ its algebraic closure. Let $K = k(x, y)$ be a separable algebraic extension of $k(x)$; $K_n = k_n(x, y)$; $\overline{K} = \bar{k}(x, y)$. Then $(x, y) \to (x^q, y^q)$ defines a correspondence I, of degrees 1, q, on a non-singular model Γ of field $\overline{K}$; I^n is the correspondence, of degrees 1, q^n, defined by $(x, y) \to (x^{q^n}, y^{q^n})$; and $I \cdot I' = q \cdot \Delta$ (more generally, any correspondence C, such that $r(C) = 1$, satisfies $C \cdot C' = s(C) \cdot \Delta$). Let ι be the element of $\mathfrak{R}$ which corresponds to I; we have $\iota \cdot \iota' = q$. The intersections of I^n with Δ, which are all found to be of multiplicity 1, are those points of Δ which have coördinates in k_n, so that the number ν_n of such points (i.e., of points on Γ with coördinates in k_n) is $f(I^n) = 1 + q^n - \sigma(\iota^n)$. But the numerator of the zeta-function $\zeta_K(s)$ of K is[6] a polynomial $P(u) = u^{2g} - (1 + q - \nu_1)u^{2g-1} + \ldots$, of degree $2g$, in $u = q^{-s}$; and, putting $P(u) = \prod_i (u - \alpha_i)$, the numerator of the zeta-function of K_n is

$$P_n(u^n) = \prod_i (u^n - \alpha_i^n) = u^{2ng} - (1 + q^n - \nu_n)u^{n(2g-1)} + \ldots,$$

so that we find that $\Sigma\alpha_i^n = 1 + q^n - \nu_n = \sigma(\iota^n)$.

Now, let $F(x) = \Sigma a_\mu x^\mu$ be any polynomial with coefficients in Z; apply $\sigma(\gamma\gamma') \geqslant 0$ to $\gamma = F(\iota)$; using $\iota \cdot \iota' = q$, $\sigma(1) = 2g$, and $\sigma(\iota'^n) = \sigma(\iota^n) = \Sigma\alpha_i^n$, we find

$$\sum_\mu a_\mu{}^2 \cdot 2g + 2 \sum_{\mu<\nu} a_\mu a_\nu q^\mu \cdot \sum_i \alpha_i^{\nu-\mu} \geqslant 0.$$

The functional equation of $\zeta_K(s)$ implies that to every root α_i of $P(u)$ there is a root $\alpha_i' = q/\alpha_i$, so that our inequality can be written as $\Sigma F(\alpha_i)F(\alpha_i') \geqslant 0$. The left-hand side of this is a quadratic form in the coefficients of F, which is $\geqslant 0$ for all integral values, and therefore also for all real values, of these coefficients. If α_1 is such that $\alpha_1\bar{\alpha}_1 = |\alpha_1|^2 \neq q$, i.e., $\alpha_1' \neq \bar{\alpha}_1$, suppose, first, that $\alpha_1' \neq \alpha_1$; put $\alpha_2 = \alpha_1'$; a polynomial $F(x)$ can be found, with real-valued coefficients, which vanishes for all roots of $P(u)$ except α_1, α_2, $\bar{\alpha}_1$, $\bar{\alpha}_2$, and takes prescribed values z_1, z_2 at α_1, α_2: then $\Sigma F(\alpha_i)F(\alpha_i') = z_1z_2 + \bar{z}_1\bar{z}_2$, which becomes < 0 for suitable z_1, z_2: this contradicts our previous inequality. We reason similarly if $\alpha_1' = \alpha_1$. Thus all roots of $P(u)$ must satisfy $|\alpha_i|^2 = q$, which is the Riemann hypothesis for ζ_K.

A detailed account of this theory, including the application to Artin's L-functions, and of the "transcendental" theory as outlined in my previous note, is being prepared for publication.

[1] Sur les fonctions algébriques à corps de constantes fini, C.R.t. 210 (1940), p. 592.

[2] F. Severi, *Trattato di Geometria algebrica*, vol. 1, pt. 1, Bologna, Zanichelli 1926, chapter VI. It should be observed that Severi's treatment, although undoubtedly containing all the essential elements for the solution of the problems it purports to solve, is meant to cover only the classical case where the field of constants is that of complex numbers, and doubts may be raised as to its applicability to more general cases, especially to characteristic $p \neq 0$. A rewriting of the whole theory, covering such cases, is therefore a necessary preliminary to the applications we have in view.

[3] A generic point of an irreducible variety of dimension n is a point, the coördinates of which satisfy the equations of the variety and generate a field of degree of transcendency n over the field of constants. Cf. B. L. van der Waerden, *Einführung in die algebraische Geometrie*, Berlin, Springer 1939, chap. IV, § 29.

[4] B. L. van der Waerden, "Zur algebraischen Geometrie XIII," *Math. Ann.*, **115**, 359, and XIV, *Ibid.* 619.

[5] Loc. cit.,[2] chap. VI, No. 75 (pp. 228–229) and No. 84 (pp. 259–267). Severi's treatment can be somewhat clarified and simplified at this point, as will be shown elsewhere.

[6] H. Hasse, "Über die Kongruenzzetafunktionen," *Sitz.-ber. d. Preuss. Akad. d. Wiss.* **1934**, 250.

THE GROUP RING OF A LOCALLY COMPACT GROUP. I

BY I. E. SEGAL[1]

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

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1. The theory of topological groups has an extensive development in the cases of locally compact[2] (l. c.) Abelian groups and compact groups. The theory of almost periodic functions on groups has proved helpful in investigating these groups, and is of interest in itself. However, there exist l. c. groups on which there are defined no almost periodic functions except for constant functions. It can therefore not be expected that these functions will be useful in investigating general l. c. groups.

In §2 we define for any l. c. group G a "group ring" $R(G)$. If G is finite $R(G)$ is the usual group ring over the field of complex numbers. $R(G)$ is defined in terms of Haar measure on G. It is known that there exists on any l. c. group a Haar measure, and this assures the existence of a non-trivial group ring. In this note we present an account of the ideal theory of a group ring for the case in which the group is either l. c. Abelian or compact. However, the conclusions of our principal theorems (Theorems 1 and 2) have meaning, and may well be true, in the case that the group in question is an arbitrary l. c. group.

The ideal theory of $R(G)$ appears to be of interest in itself, apart from its relation to the structure of G. In the particular case in which G is the additive group of all real numbers in its usual topology this theory plays a fundamental rôle in some aspects of Fourier analysis.

2. On any l. c. group G there exists a right invariant Haar measure.[3] This is a completely additive, non-negative set function μ, defined on a complete σ-field $\mathfrak{F}$ which includes the Borel subsets of G, and possessing certain special properties. We shall not discuss these properties except to note that $\mu(Sa) = \mu(S)$ for every a in G and S in $\mathfrak{F}$. (Sa means the set of all elements of G of the form xa, with $x \in S$.)

Let p satisfy $1 \leq p < \infty$. Let $L_p(G)$ be the space of all functions $f(x)$ on G to the complex numbers which are measurable (μ) and such that $\int_G |f(x)|^p d\mu(S_x)$ is finite. When $p = 1$ we write simply $L(G)$. $L_p(G)$ is a Banach space if one defines the norm by $||f||_p = [\int_G |f(x)|^p d\mu(S_x)]^{1/p}$. If f and g are generic elements of $L_p(G)$ and $L(G)$, respectively, we define the convolution of f with g to be the element h of $L_p(G)$, where h is given by

$$h(x) = \int_G f(xy^{-1})g(y)d\mu(S_y).$$

The existence of this integral for almost all x, and the fact that $||h||_p \leq ||f||_p||g||_1$, can be established by using Fubini's theorem. There are tech-

nical difficulties here which are overcome by making use of special properties of Haar measure.

It is now easy to see that $L(G)$ is a ring if the product of two elements is defined to be their convolution. Let $L^1(G)$ be the class of all ordered pairs

$$(f, \alpha), \tag{2.1}$$

with $f \in L(G)$ and α a complex number. With the definitions $\beta(f, \alpha) = (\beta f, \beta\alpha)$, $(f, \alpha) + (g, \beta) = (f + g, \alpha + \beta)$, $(f, \alpha)(g, \beta) = (fg + \alpha g + \beta f, \alpha\beta)$, $L^1(G)$ is a ring. We define $\rho = 0$ or 1 according as $L(G)$ has or does not have a principal unit; either case may occur. The set $L^2(G)$ of all elements of $L^1(G)$ of the form (2.1) with $\alpha = \alpha\rho$ is a ring with a principal unit. We call $L^2(G)$ the *group ring* of G and write it $R(G)$. Let $L^3(G)$ be the subring of $R(G)$ which consists of all elements of the form (2.1) with $\alpha = 0$. There is an obvious isomorphism between $L^3(G)$ and $L(G)$.

3. In this section R is an arbitrary ring with a principal unit and scalar multiplication by complex numbers. By an "ideal" in R we mean a two-sided ideal which is also a linear subspace in R. An ideal M is "maximal" if (1) $M \neq R$, and (2) if I is an ideal such that $M \subset I$, then either $M = I$ or $R = I$. Let $\mathfrak{M}$ be the family of all maximal ideals of R. We topologize $\mathfrak{M}$ by defining the closure $\overline{\mathfrak{N}}$ of a generic subset $\mathfrak{N}$ of $\mathfrak{M}$ as follows: $M_0 \in \overline{\mathfrak{N}}$ means $M_0 \supset \Pi M$, where the intersection is taken over all M in $\mathfrak{N}$. It can be shown that in this topology $\mathfrak{M}$ is a compact T_1-space.[4]

We shall call the ring R *semi-simple*[5] in case the intersection of all maximal ideals in R consists only of the zero element. R is said to be simple if it contains no proper ideal except the null ideal. If M is a maximal ideal in R, the (simple) quotient ring R/M is called a simple map of R.

4. THEOREM 1. *Let G be a locally compact group which is either compact or Abelian. Then the group ring of G is semi-simple.*[6]

In the Abelian case the proof rests on (1) the observation that (in this case) $R(G)$ is commutative, (2) the theorem that every proper ideal in $R(G)$ is contained in a maximal ideal, (3) the theorem that a bounded self-adjoint operator in Hilbert space with no point in its spectrum except zero is identically zero. In the compact case the proof rests on (1) the observation that (in this case) every simple map of $R(G)$ is an algebra of finite order, (2) the structure theorems of Wedderburn,[7] (3) the theory of completely continuous operators in Hilbert space, and (4) the theorem that every bounded representation of a group is equivalent to a unitary representation.[8]

THEOREM 2. *Let G be a locally compact group which is either compact or Abelian. Let K be a proper ideal in the group ring $R(G)$. Let $\mathfrak{N}$ be the family of all maximal ideals in $R(G)$ which contain K. Let $\mathfrak{O}$ be any family of maximal ideals such that $\mathfrak{O}$ is open (in the topology of §3) and $\mathfrak{O} \supset \mathfrak{N}$. Then*

$$\Pi_1 M \supset K \supset \Pi_2 M,$$

where $\Pi_1 M$ is the intersection of all ideals in $\mathfrak{R}$, and $\Pi_2 M$ is the intersection of all ideals in $\mathfrak{D}$.

Thus every ideal is in a sense approximately equal to the intersection of the maximal ideals which contain it. The proof depends on (1) Theorem 1, (2) the observation that the operation * on $R(G)$ to $R(G)$ defined by $(f, \alpha)^* = (f^*, \bar{\alpha})$, where $f^*(x) = \overline{f(x^{-1})}$ has, among other properties, the property that if M is a maximal ideal in $R(G)$, and if $x_i \in R(G)$ and $\Sigma_{i=1}^n x_i x_i^* \in M$, then $x_i \in M$ $(i = 1, 2, \ldots, n)$. The observation (2) is a consequence of Theorem 3 (which does not depend on Theorem 1 or Theorem 2).

5. Theorems 3, 4 and 5 amplify Theorems 1 and 2.

THEOREM 3. *Let G be a locally compact group which is either compact or Abelian. If G is discrete, $L^3(G) = R(G)$. If G is not discrete, $L^3(G)$ is a maximal ideal in $R(G)$. Let M be any maximal ideal in $R(G)$ such that M is different from $L^3(G)$. Then*

(1) There is a bounded continuous irreducible representation $g \to m(g)$ of G as matrices of some finite order n, such that the mapping

$$(f, \alpha\rho) \to \alpha\rho e_n + \int_G m(g) f(g) d\mu(S_g)$$

on $R(G)$ to the ring C_n of all complex matrices of order n is an exhaustive homomorphism. Here e_n is the unit matrix of order n, and the integrand is a matrix.

(2) The mapping given in (1) is continuous on $R(G)$ to C_n with the topology on $R(G)$ that given by the norm

$$||(f, \alpha)|| = ||f||_1 + |\alpha|$$

and the topology on C_n the usual one.

(3) $(f, \alpha) \in M$ if and only if $(f, \alpha) \to 0$. Furthermore, there corresponds to any bounded continuous irreducible representation of G as matrices of finite order a maximal ideal M in $R(G)$ such that (1), (2) and (3) hold. In case G is Abelian the degree n of the representation described in (1) is unity. Finally, the correspondence is such that to different maximal ideals there correspond inequivalent representations, and conversely.

The proof depends on (1) a theorem on the form of linear functions on the space $L(G)$, (2) a structure theorem of Wedderburn,[7] (3) results in the theory of Banach spaces.

6. THEOREM 4. *Let G be a compact group. Let $\mathfrak{M}$ be the family of all maximal ideals in $R(G)$. The topology on $\mathfrak{M}$ defined in §3 may be described as follows.*

In case G is finite, $\mathfrak{M}$ is discrete. In case G is infinite, $L^3(G)$ is a maximal ideal in $R(G)$, and a generic neighborhood of $L^3(G)$ is the set $\mathfrak{M}$-$\mathfrak{N}$, where $\mathfrak{N}$ is an arbitrary finite subset of $\mathfrak{M}$ which does not contain $L^3(G)$. If M is a

maximal ideal in $R(G)$ such that M is different from $L^3(G)$, then the set consisting only of the point M is an open set in $\mathfrak{M}$.

The theorem follows from the orthogonality of inequivalent irreducible representations of G.

If G is a l. c. Abelian group we denote the group of all continuous characters of G by G^*. When G^* is topologized in the way described by van Kampen[9] we write it $\bar{G}$. Since a character of an Abelian group is a bounded irreducible representation, Theorem 3 describes a correspondence between elements of $\bar{G}$ and elements of $\mathfrak{M}$. Denote this correspondence by $x \to \varphi(x)$ ($x \in \bar{G}$, $\varphi(x) \in \mathfrak{M}$). This correspondence exhausts $\mathfrak{M}$ except possibly for $L^3(G)$, which may be in $\mathfrak{M}$, but which does not correspond to any character.

THEOREM 5. *Let G be a locally compact Abelian group. Let $\mathfrak{M}$ be the family of all maximal ideals of the group ring $R(G)$. Let $\mathfrak{M}'$ be $\mathfrak{M}$ or $\mathfrak{M} - \{L^3(G)\}$ according as G is discrete or is not discrete. Let $x \to \varphi(x)$ denote the one-to-one exhaustive correspondence between $\bar{G}$ and $\mathfrak{M}'$ described above. $\mathfrak{M}'$ is to have the relative topology induced by the topology on $\mathfrak{M}$. Then the mapping $x \to \varphi(x)$ is a homeomorphism.*

Theorem 5 follows from a decomposition theorem for l. c. Abelian groups proved by van Kampen.[9]

7. THEOREM 6. *Let G be a compact group. Let K be a closed proper ideal in the group ring $R(G)$. Then K is equal to the intersection of all maximal ideals which contain it.*

The proof depends on the above results and on the theory of almost periodic functions on groups.[8] We may note here that the group ring of a compact group is regular in the sense of von Neumann. In general, however, a group ring is not regular.

8. The above results have applications in a variety of fields. In particular they seem to throw some light on the theory of almost periodic functions on groups and on certain kinds of Tauberian theorems. Theorem 1 implies the following theorems on almost periodic functions: (1) any l. c. Abelian group is maximally almost periodic; (2) every almost periodic function on a group can be uniformly approximated by linear combinations of coördinates of irreducible representations.[10] Theorem 2, for the case in which G is the additive group of all real numbers in its usual topology, is an extension of theorems of a Tauberian nature due to Wiener[11] and to Pitt.[12] We hope to discuss these and other applications more fully in a later note. Finally, we also hope to be able to discuss the situation in a general l. c. group.

[1] The results presented in this note are an extension of some of the results in a doctoral dissertation presented to the faculty of Yale University in April, 1940.

[2] We use "compact" in the same sense in which "bikompakt" is used in Alexandroff and Hopf, *Topologie*, Berlin, 1936.

[3] The existence of Haar measure of the generality here required was first indicated by A. Weil, *Compt. rend.*, Paris, **202**, 1147–1149 (1936).

[4] This topology is similar to one used by H. Wallman, *Ann. Math.* (s. 2) **39**, 112–126 (1938).

[5] In case R has a finite basis this notion of semi-simplicity is equivalent to the usual one.

[6] Theorem 1 is well known in the case that G is finite. It was first proved by H. Maschke, *Math. Ann.*, **52**, 363–368 (1899). The part of Theorem 3 which refers to Abelian groups and a theorem equivalent to the part of Theorem 1 which refers to Abelian groups are stated by I. Gelfand and D. Raikov, *Compt. rend. acad. sci. U. R. S. S.*, **28**, 195–198 (1940). The proofs they indicate are different from ours.

[7] J. H. M. Wedderburn, *Proc. Lond. Math. Soc.*, (s. 2) **6**, 77–118 (1907).

[8] See J. von Neumann, *Trans. Amer. Math. Soc.*, **36**, 445–492 (1934).

[9] See E. R. van Kampen, *Ann. Math.*, (s. 2) **36**, 448–463 (1935).

[10] Theorem 1, for the case of a separable group, and Theorem 2 were first proved by J. von Neumann in the paper cited in footnote 7 by methods quite different from those used here. Theorem 1 also follows from results in the papers cited in footnotes 6 and 9.

[11] N. Wiener, *Ann. Math.*, (s. 2) **33**, 1–100 (1932) Theorem III.

[12] H. R. Pitt, *Jour. Lond. Math. Soc.*, **15**, 97–112 (1940) Theorem 2.

CONDITIONAL INVARIANTS

By G. Y. Rainich

Department of Mathematics, University of Michigan

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1. If we have a plane curve referred to cartesian coördinates and represented parametrically with arc length as parameter by $x = f(s)$, $y = g(s)$, the expression for the curvature

$$\gamma = \dot{x}\,\ddot{y} - \dot{y}\,\ddot{x} \tag{1}$$

may also be written as

$$\dddot{y}/\dot{x} \tag{2}$$

where a dot over a letter indicates differentiation with respect to arc length. To prove it we notice first that

$$\dot{x}^2 + \dot{y}^2 = 1, \qquad \dot{x}\,\ddot{x} + \dot{y}\,\ddot{y} = 0. \tag{3}$$

Using these relations we can write

$$-\frac{\ddot{x}}{\dot{y}} = \frac{\ddot{y}}{\dot{x}} = -\frac{\ddot{x}\,\dot{y}}{\dot{y}^2} = \frac{\ddot{y}\,\dot{x}}{\dot{x}^2} = \frac{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}{\dot{x}^2 + \dot{y}^2} = \dot{x}\,\ddot{y} - \dot{y}\,\ddot{x} = \gamma.$$

The expressions (1) and (2) represent the same quantity but their structure is different. The first is invariant under rotations as is apparent from

its form in virtue of the fact that $(\dot{x}, \dot{y})$ and $(\ddot{x}, \ddot{y})$ are components of vectors; the second is "not invariant in form;" in order to prove its invariance we have to take into account conditions (3). We shall call such expressions *conditional invariants*, and we shall say in general:

Definition 1. An expression $F(a)$ is called a conditional invariant modulo R under a transformation T if $F(Ta) = F(a)$ for those a for which R hold. Here a stands for a set of variables, Ta for the transformed set, $F(a)$ for an expression or set of expressions and R for a relation or relations. In the example given above, a stands for the vectors $(\dot{x}, \dot{y})$ and $(\ddot{x}, \ddot{y})$, $F(a)$ stands for (2) and R stands for (3). All the transformations considered in this paper are Euclidean or pseudo-Euclidean rotations.

2. In the above example it was possible to find an ordinary, *non-conditional* invariant, viz., $\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}$ which has the same value as the conditional invariant $\ddot{y}/\dot{x}$ modulo relations (3). The question arises whether this is always the case. The following examples show that this is not so.

The simplest example is given by two collinear vectors with zero squares in a complex space or in a real pseudo-Euclidean space. This system possesses an invariant, viz., the scalar σ by which we must multiply one vector to obtain the other; but every ordinary invariant of two vectors is a function of their squares and their scalar product, and here all these quantities are zero. Therefore σ is not equal to any ordinary invariant, but it is a conditional invariant modulo the relations saying that the squares of the two vectors and their scalar product are equal to zero. Such an invariant we shall call a *singular conditional* invariant. In general:

Definition 2. We shall call a conditional invariant modulo R a singular conditional invariant if there does not exist an ordinary invariant to which it is equal modulo R.

For a more specific discussion consider a pseudo-Euclidean three-dimensional space and in it two vectors of components (u, v, w) and (p, q, z). Subject them to the conditions

$$u^2 + v^2 - w^2 = 0, \qquad p^2 + q^2 - r^2 = 0, \qquad up + vq - wr = 0. \tag{4}$$

Then we can easily prove that, unless $r = 0$, there exists a number σ such that

$$u = p\,\sigma, \qquad v = q\,\sigma, \qquad w = r\,\sigma.$$

This number is an invariant as one sees immediately observing that the transformation formulas are linear and homogeneous.

3. One might think that a singular conditional invariant, which by definition is not *equal* to any ordinary invariant, can be obtained as a *limit* of such an invariant, that, for instance, in the case of the conditional

invariant σ in the above example there may exist a function $f(u, v, w, p, q, r)$ which is invariant and which tends toward the limit σ when u, v, w, p, q, r tend toward values which satisfy the relations (4). It is not difficult to prove that such a function does not exist. Of course, it is possible to obtain σ as a limiting value, for instance, of the ordinary invariant $\frac{u^2 + v^2 - w^2}{p^2 + q^2 - r^2}$; if we start with $u_0\, v_0\, w_0\, p_0\, q_0\, r_0$ which satisfy relations (4), set $u = u_0 + \epsilon$, $v = v_0$, $w = w_0$; $p = p_0 + \eta$, $q = q_0$, $r = r_0$ and make ϵ and η go to zero so that $\epsilon/\eta \to 1$, the limit will be exactly u_0/p_0, that is σ; but for other passages to the limit, for instance, one in which we make $\epsilon/\eta \to 2$ we would get a different result.

4. Every singular conditional invariant I know may be written in several ways as fractions with different denominators; to give an example consider a vector of (complex) components (u, v, w) such that

$$u^2 + v^2 + w^2 = 0. \tag{5}$$

Let u, v, w depend on a parameter and denote differentiation with respect to that parameter by a prime. Then

$$uu' + vv' + ww' = 0.$$

Now notice that in virtue of these relationships

$$\frac{1}{u}\begin{vmatrix} v' & w' \\ v & w \end{vmatrix} = \frac{1}{v}\begin{vmatrix} w' & u' \\ w & u \end{vmatrix} = \frac{1}{w}\begin{vmatrix} u' & v' \\ u & v \end{vmatrix}.$$

To prove the first equality we have

$$\frac{1}{u}\begin{vmatrix} v' & w' \\ v & w \end{vmatrix} = \frac{1}{uv}\begin{vmatrix} vv' & w' \\ v^2 & w \end{vmatrix} = \frac{1}{uv}\begin{vmatrix} vv' + ww' & w' \\ v^2 + w^2 & w \end{vmatrix}$$

$$= \frac{1}{uv}\begin{vmatrix} -uu' & w' \\ -u^2 & w \end{vmatrix} = \frac{1}{v}\begin{vmatrix} w' & u' \\ w & u \end{vmatrix}.$$

It is now easy to prove that the common value of (5) is an invariant. Since a general rotation may be obtained as a result of three rotations each involving only two of the coördinates, any expression which is not affected by any rotation involving two coördinates is invariant; now if we want to show that (5) is not affected e.g., by a x–y rotation we choose the third form; the denominator there is not affected and neither is the numerator and this proves the invariance. If instead of differentiating with respect to a parameter we differentiate with respect to the coördinates x, y and z we obtain three expressions which transform like the components of a vector. This gives an example of a conditional *covariant*.

If instead of a vector in three dimensions we start with one in four dimensions satisfying the relation

$$u^2 + v^2 + w^2 + s^2 = 0$$

expressions of the form

$$\frac{1}{s}\begin{vmatrix} u & v & w \\ u_x & v_x & w_x \\ u_y & v_y & w_z \end{vmatrix} \tag{6}$$

furnish six components (each capable of being represented in four forms) of an alternating tensor or six vector.

5. The fact that what we call ordinary invariants cease to furnish *all* invariants when the components of vectors cease to be independent has, of course, been noticed before, for instance, explicitly by E. Study.[1] The trouble arising from this, or more specifically from the vanishing of the square of certain vectors has been remedied in various ways[1,2] but conditional invariants do not seem to have been introduced before.

The idea of introducing them arose in connection with the study of Dirac equations; in particular, invariants of type (6) first appeared in the work of Gordon Fuller devoted to a special case of the Dirac equations.[3] The application of conditional invariants to the general case of Dirac equations is reserved for another paper.

[1] Cf. E. Study, "Zur Differentialgeometrie der analytischen Curven," *Trans. A. M. S.* **10** (1909), footnote on p. 9.

[2] For instance, J. Lense, "Ueber die Ableitungsgleichungen der ametrischen Mannigfaltigkeiten," *Math. Z. S.*, **34,** 721–36 (1930).

[3] To be published elsewhere.

THE DIRAC EQUATIONS AND CONDITIONAL INVARIANTS

BY G. Y. RAINICH

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN

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1. We write the Dirac equations for the electron in the absence of field as[1]

$$\begin{aligned} -d_x + id_y - c_z + c_t + ika &= 0 \\ -c_x - ic_y + d_z + d_t + ikb &= 0 \\ b_x - ib_y + a_z + a_t + ikc &= 0 \\ a_x + ia_y - b_z + b_t + ikd &= 0. \end{aligned} \tag{1}$$

Subscripts indicate partial differentiation.

These equations although in a sense invariant under Lorentz transformations do not appear in tensor form, and the considerations presented in this paper arise from the desire to replace the above equations by an equivalent set expressed in tensor form.[2] This can be done by introducing instead of the quantities a, b, c, d certain quadratic expressions of these quantities. Such quadratic expressions may be obtained in the following way: The above differential equations may be written in the form

$$\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}+C\frac{\partial}{\partial z}+D\frac{\partial}{\partial t}+ik\,I\right)V=0,$$

where A, B, C, D are four-row matrices. *One way* to obtain quadratic expressions is to consider these matrices and their products (altogether, counting the identity, 16 matrices) as matrices of coefficients of Hermitian forms of the variables a, b, c, d. The expressions so obtained were introduced by Darwin.[3] But there is also another way: of the sixteen matrices, six are skew symmetric and ten are symmetric. The symmetric ones may be used not only as coefficients of Hermitian forms but also as coefficients of *quadratic forms* and so they give rise to ten quantities which we shall write as

$$\begin{aligned}&X=-i(a^2-b^2+c^2-d^2),\ Y=a^2+b^2+c^2+d^2,\quad Z=2i(ab+cd)\\&L=a^2-b^2-c^2+d^2,\qquad M=i(a^2+b^2-c^2-d^2),\ N=2(-ab+cd)\\&s=2(ad-bc),\ u=2(-ac+bd),\ v=-2i(ac+bd),\ w=2(ad+bc).\end{aligned}$$

Some of these expressions were used by Whittaker[4] and we shall say that they are of Whittaker type. Of course, being obtained from four quantities, these ten are not independent but satisfy many relations which we call R and of which we will write out only a few as examples:

$$(R)\qquad \begin{aligned}&X^2+Y^2+Z^2=L^2+M^2+N^2=u^2+v^2+w^2=s^2\\&XL+YM+ZN=0,\ Xu+Yv+Zw=0,\ Lu+Mv+Nw=0.\end{aligned}$$

2. Using these expressions it is possible to write the equations (1) as

$$\begin{aligned}&Y_z-Z_y+L_t=F_{(x)} \qquad M_z-N_y-X_t+2ku=G_{(x)}\\&Z_x-X_z+M_t=F_{(y)} \qquad N_x-L_z-Y_t+2kv=G_{(y)}\\&X_y-Y_x+N_t=F_{(z)} \qquad L_y-M_x-Z_t+2kw=G_{(z)}\\&L_x+M_y+N_z=F_{(t)} \qquad X_x+Y_y+Z_z+2ks=G_{(t)}.\end{aligned}\tag{2}$$

It may be noticed that the left-hand sides of these equations are the same as those of Maxwell's equations, but in Maxwell's equations the right-hand sides are zero. Here these right-hand sides are the components of two vectors which are *conditional covariants* modulo the relations R; the conception of conditional covariance has been introduced in another paper.[4]

More precisely the right-hand sides of (2) are obtained as follows. Using relations R one proves that the expressions

$$F = \frac{1}{s^2}\begin{vmatrix} L' & M' & N' \\ L & M & N \\ X & Y & Z \end{vmatrix} \text{ and } G = \frac{1}{s^2}\begin{vmatrix} X' & Y' & Z' \\ X & Y & Z \\ L & M & N \end{vmatrix},$$

where primes indicate differentiation with respect to a parameter, are invariant under the transformations of the Lorentz group; in other words, they are conditional invariants[4] modulo R. If, however, instead of differentiating with respect to a parameter we differentiate with respect to four coördinates x, y, z, t we obtain for each four expressions which are the components of a covariant vector. It is these components that we denote by $F_{(x)}$, etc., in the equations (2); the subscripts indicate that they have been obtained by differentiating with respect to the corresponding coördinates and parentheses around the subscripts are intended to show that the quantities are *not* simply derivatives with respect to these coördinates.

The invariants F and G, as is usually the case with conditional invariants, are capable of being represented in many different forms. We give as an example the formulas

$$F = \frac{L's - Y'w + Z'v}{u} = \frac{L'u + M'v + N'w}{s} = \frac{L'Z - N'X + s'v}{w} \quad (3)$$

and mention that G as well as F may be written as a fraction with any one of the ten quantities $u, v, w, s, X, Y, Z, L, M, N$ as denominator.

3. Altogether sixty expressions analogous to (3) may be written; ten of them give F, another ten give G and the remaining forty fall into four groups of ten each giving four components P, Q, R, S of a vector. For P, for instance, we have

$$\frac{L'u + v'M + w'N}{X} = \frac{-LZ' + XN' - vs'}{w} = \frac{u'Z - M's - X'w}{N}$$

and seven more of that type.

With the use of P, Q, R, T we can write another set of equations which are also equivalent to the Dirac equations, viz.,

$$\begin{aligned} s_x + u_t - 2kX &= R_{(y)} - Q_{(z)} & v_z - w_y + 2kL &= T_{(x)} + P_{(t)} \\ s_z + v_t - 2kY &= P_{(z)} - R_{(x)} & w_x - u_z + 2kM &= T_{(y)} + Q_{(t)} \\ s_z + w_t - 2kZ &= Q_{(x)} - P_{(y)} & u_y - v_x + 2kN &= T_{(z)} + R_{(t)} \end{aligned}$$

where the subscripts in parentheses have the same meaning as explained in *2*.

This gives another example of the richness of the field of conditional invariants in different forms of one and the same thing. That might be considered as a disadvantage in some cases, but it certainly is an advantage in dealing with fractions of the form (3); since it is impossible to do away with

fractional expressions at all, the possibility to choose your denominator permits more ease in handling expressions of this type.

4. As was stated in *1*, there exist other quadratic expressions which behave like tensor components and they also might be used in obtaining a system of tensor equations equivalent to the Dirac equations. These expressions possess what is called gauge invariance; that is, they do not change when we multiply a, b, c, d by $e^{i\alpha}$; this may give them an advantage from the point of view of physics, but as a consequence a, b, c, d cannot be expressed in terms of the Darwin quantities and the situation becomes more complicated; the general case has not been treated as far as I know. However, in the special case when in equations (1)

$$k = 0, \quad \bar{a} = d, \quad \bar{b} + \bar{c} = 0$$

Gordon Fuller has derived a tensor form system of equations in terms of Darwin type quantities. In doing this he was led to another type of conditional invariants of which an example is indicated in.[5]

[1] Cf. H. Weyl, *Gruppentheorie und Quantenmechanik*, Leipzig, 1928, p. 171; we denote $\psi_1, \psi_2, \psi_3, \psi_4$ by a, b, c, d, respectively.

[2] Cf. the author's "Spinors and Tensors," *Bull. Am. Math. Soc.*, **42,** 104 (1936).

[3] C. G. Darwin, *Proc. Roy. Soc.* London Series A, **120,** 621–31 (1928).

[4] E. T. Whittaker, "On the Relation of the Tensor-Calculus to the Spinor-Calculus," *Ibid.*, **158,** 38–46 (1937).

[5] "Conditional Invariants," these PROCEEDINGS, **27,** 352 (1941).